Lecture Notes for the course
"Design and Operation of Traffic and Telecommunication Networks"

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Bachelor of Science in Mathematics
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by

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1 "Corner" points of a polyhedron

In this section, we provide three alternative characterizations of "corner" points of a polyhedron. These assume a special role in linear programming, since, informally speaking, we can say that an optimal solution of a linear program "tends" to correspond to one of such points.

The first characterization of "corner" point, refers to the fact that these points cannot be expressed as a convex combination of other two points of the polyhedron,

Definition (extreme point): let $P \subseteq \mathbb{R}^n$ be a polyhedron. A vector $x \in P$ is an *extreme point* of P if there are no two vectors $y, z \in P$: $y \neq x$, $z \neq x$ and a scalar $\lambda : 0 \leq \lambda \leq 1$ such that:

$$x = \lambda y + (1 - \lambda) z \ .$$

The second characterization refers to the "corner" point as to the unique optimal solution of a linear program having P as feasible set.

Definition (vertex): Let P be a polyhedron. A vector $x \in P$ is a *vertex* of P if there exists some vector $c \in \mathbb{R}^n$: $c'x < c'y$ for every $y \in P$: $y \neq x$.

The third characterization refers to the "corner" point in terms of a set of linear constraints and is introduced since it is particularly useful from an algorithmic point of view.
As first step, we introduce a polyhedron $P \subseteq \mathbb{R}^n$ defined by the following systems of linear equality and inequality constraints:

$$\begin{aligned} a_i'x &\geq b_i & i \in I_1 \\ a_i'x &\leq b_i & i \in I_2 \\ a_i'x &= b_i & i \in I_3 \end{aligned}$$

We say that a constraint i belonging to the previous system is *active* in $\bar{x}$ if $a_i'\bar{x} = b_i$.
Given these premises, the following result holds:

Theorem: Let $\bar{x} \in \mathbb{R}^n$ and let $I^{ACT} = \{i \in I : a_i'\bar{x} = b_i\}$ be the set of indices of constraints that are active in $\bar{x}$. The following statements are equivalent:

1. there exist n vectors in the set $\{a_i : i \in I^{ACT}\}$ that are linearly independent;
2. the *span* of the vectors in the set $\{a_i : i \in I^{ACT}\}$ is $\mathbb{R}^n$;
3. the system of equations $a_i'\bar{x} = b_i$ with $i \in I^{ACT}$ has a unique solution.

After having introduced such result, we can proceed to give the third characterization of corner point as a point of the polyhedron where there are n active constraints corresponding with linearly independent vectors a_i.

Definition (basic feasible solution): let P be a polyhedron.
A vector $\bar{x} \in \mathbb{R}^n$ is a *basic solution* if: a) all equality constraints are active; b) among the vectors a_i associated with constraints active in $\bar{x}$, there are n vectors that are linearly independent.
A vector $\bar{x} \in \mathbb{R}^n$ that is a *basic solution* and that additionally satisfies all the constraints defining P is a *basic feasible solution.*

Theorem: Let P be a non-empty polyhedron and $\bar{x} \in P$. The following statements are equivalent:

1. $\bar{x}$ is an extreme point;
2. $\bar{x}$ is a vertex;
3. $\bar{x}$ is a basic feasible solution.

Proof. We prove the statement following the implication order $2 \Rightarrow 1$, $1 \Rightarrow 3$, $3 \Rightarrow 2$. Note that without loss of generality we assume that P is defined only by constraints of the type $a_i'x \geq b_i$ and $a_i'x = b_i$.

vertex $\Rightarrow$ extreme point
Suppose that $\bar{x}$ is a vertex and consider any two points $y, z \in P$: $y \neq \bar{x}$ and $z \neq \bar{x}$. Consider additionally a scalar $0 \leq \lambda \leq 1$.
By definition of vertex, there exists $c \in \mathbb{R}^n : c'\bar{x} \leq c'y$ and $c'\bar{x} \leq c'z$. This implies that $c'\bar{x} \leq c'(\lambda y + (1-\lambda)z)$ and then that $\bar{x} \neq \lambda y + (1-\lambda)z$, thus showing that $\bar{x}$ cannot be expressed as a convex combination of other points of P and is thus an extreme point.

extreme point $\Rightarrow$ basic feasible solution
We prove this by contradiction, assuming that an extreme point $\bar{x}$ is not a basic feasible solution.
Let $I^{ACT} = \{i \in I : a_i'\bar{x} = b_i\}$. Since $\bar{x}$ is not a basic feasible solution, there are no n linearly independent vectors in $\{a_i : I^{ACT}\}$. As a consequence, the vectors a_i with $i \in I^{ACT}$ lie in a proper subspace of $\mathbb{R}^n$ and there exists a non-zero vector $d \in \mathbb{R}^n$ such that $a_i'd = 0$, for all $i \in I^{ACT}$.
Let $\epsilon > 0$ be a small number and consider the vectors $y = \bar{x} + \epsilon d$ and $z = \bar{x} - \epsilon d$. It can be noted that $a_i'y = a_i'z = a_i'\bar{x}$ for $i \in I^{ACT}$. Moreover, for $i \notin I^{ACT}$, it holds $a_i'\bar{x} > b_i$ and, for sufficiently small ϵ, it also holds $a_i'y > b_i$ (ϵ must be such that $\epsilon|a_i'd| < a_i'\bar{x} - b_i$). Therefore $y \in P$ and, through similar arguments, we can prove that $z \in P$. By finally noticing that:

$$\bar{x} = \frac{y+z}{2}$$

we obtain the contradiction that $\bar{x}$ can be expressed as a convex combination of y and z thus contradicting the fact that $\bar{x}$ is an extreme point.

basic feasible solution $\Rightarrow$ vertex
Let $\bar{x}$ be a basic feasible solution and $I^{ACT} = \{i \in I : a_i'\bar{x} = b_i\}$ be the set of indices of active constraints in $\bar{x}$.
If we define the cost vector $c = \sum_{i \in I^{ACT}} a_i$, we have:

$$c'\bar{x} = \sum_{i \in I^{ACT}} a_i\bar{x} = \sum_{i \in I^{ACT}} b_i$$

and for every $x \in P$ and i, it holds $a_i'x \geq b_i$ and

$$c'x = \sum_{i \in I^{ACT}} a_i x \geq \sum_{i \in I^{ACT}} b_i$$

The two chains of (in)equalities show that $\bar{x}$ is an optimal solution for the problem of minimizing $c'x$ over P. Additionally, in the second chain, the equality holds if and only if $a_i'x = b_i$ for every I^{ACT}.
Since $\bar{x}$ is a basic feasible solution, there are n linearly independent constraints that are active in $\bar{x}$ and $\bar{x}$ is the unique solution to the system of equations defined by $a_i'x = b_i$ with I^{ACT}, on the basis of the previous theorem. It follows that $\bar{x}$ is the unique optimal solution of $c'x$ over P and, by definition, $\bar{x}$ is a vertex.